PHYS 321
Assignment 3
Due Wednesday March 21, 2018

Read Chapter 4, 5

Problems: 4.2, 4.10, 4.14, 4.15, 4.20, 4.21, 4.26

4.2 $\rho(r) = \frac{q}{4\pi a^3} e^{-2r/a}$

Use Gauss's Law $\oint \vec{E} \cdot d\vec{a} = \frac{q}{\varepsilon_0}$

$$4\pi r^2 E(r) = \int_0^r \frac{4\pi r'^2}{\varepsilon_0} \rho(r')\, dr'$$

$$E(r) = \frac{1}{\varepsilon_0 r^2} \int_0^r dr'\, r'^2 \frac{q}{\pi a^3} e^{-2r'/a}$$

$$= \frac{q}{\pi \varepsilon_0 r^2 a^3} \int_0^r dr'\, r'^2 e^{-2r'/a}$$

Following the authors hint, let

$$e^{-2r'/a} \simeq 1 - \frac{2r'}{a}$$

$$E(r) = \frac{q}{\pi \varepsilon_0 r^2 a^3} \int_0^r dr'\, r'^2 \left(1 - \frac{2r'}{a}\right)$$

$$\int_0^r dr' \left(r'^2 - \frac{2r'^3}{a}\right) = \frac{r^3}{3} - \frac{1}{2}\frac{r^4}{a}$$

$$E(r) = \frac{q}{\pi \varepsilon_0} \frac{1}{a^3} \left(\frac{r}{3} - \frac{r^2}{2a}\right) = \frac{q}{\pi \varepsilon_0} \frac{r}{3a^3} \left(1 - \frac{r}{2a}\right)$$

Again, let $\frac{r}{a} \ll 1$

$$E(r) = \frac{q}{\pi \varepsilon_0} \frac{r}{3a^3}$$

the external field shifts the relative position of the proton and electron cloud until $E(r) = E_{ext}$.

$$\frac{q\,d}{3\pi\varepsilon_0 a^3} = \frac{p}{3\pi\varepsilon_0 a^3} = E_{ext}$$

$$p = \alpha E_{ext} \Rightarrow \alpha = 3\pi\varepsilon_0 a^3$$

4-10 $\vec{P}(\vec{r}) = k\vec{r} \quad r \leq R$

a) $\sigma_b = \vec{P}\cdot\hat{n} = kR \qquad q_b = (4\pi R^2)kR$ surface

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 k r)$$

$$= -\frac{k}{r^2}3r^2 = -3k \qquad q_b = \frac{4}{3}\pi R^3(-3k)$$

$$= -4\pi k R^3 \quad \text{volume}$$

Note surface + volume charges = 0

b) $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = 0 \qquad \vec{E} = -\frac{\vec{P}}{\varepsilon_0} = -\frac{k\vec{r}}{\varepsilon_0}$ for $r < R$

For $r > R \quad \vec{P} = 0,\ \vec{D} = 0,\ \vec{E} = 0$

4.14 $\sigma_b = \vec{P}\cdot\hat{n} \quad \rho_b = -\nabla\cdot\vec{P}$

$$Q_{tot} = \oint \sigma_b \, da + \int \rho_b \, d\tau$$

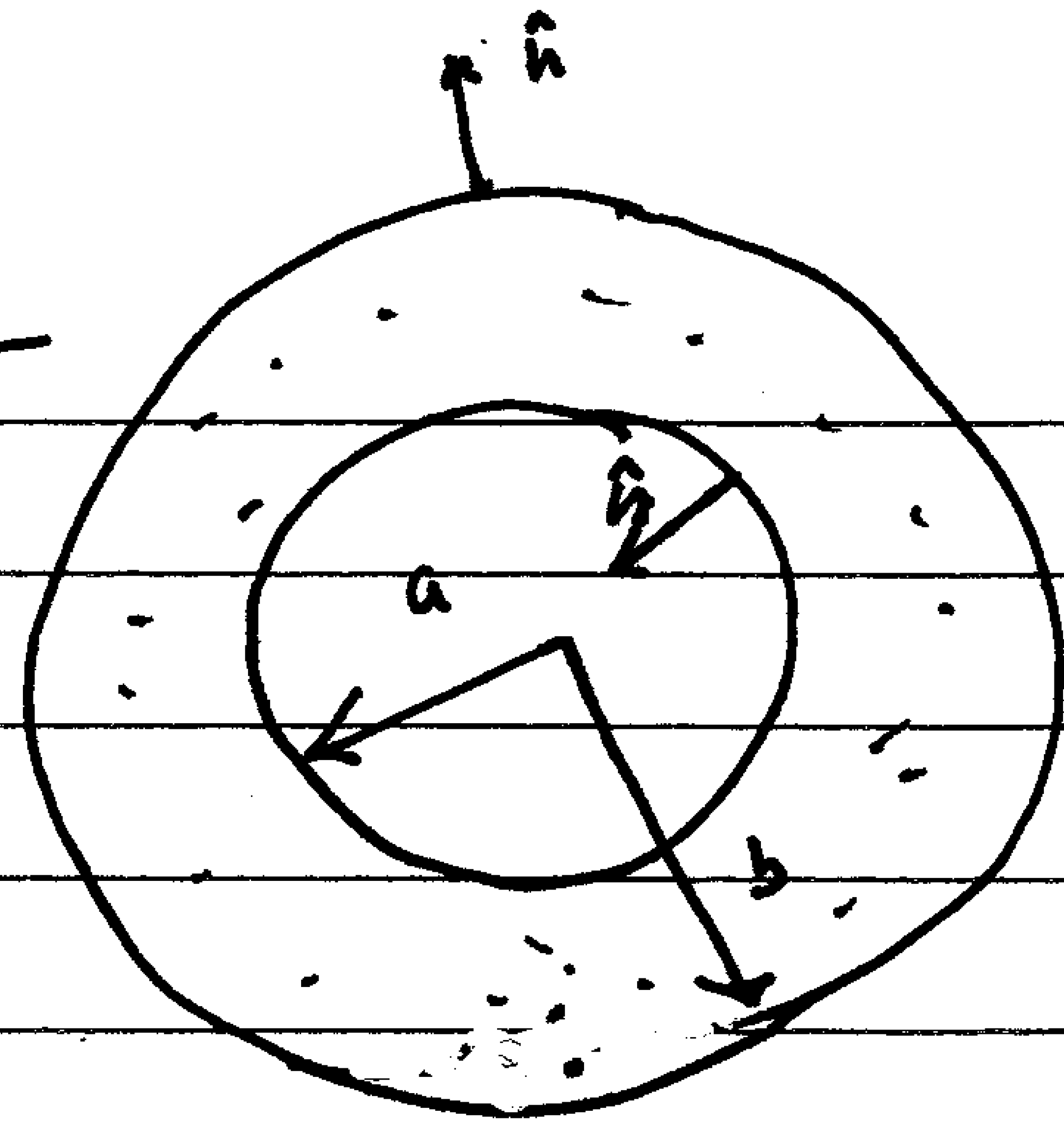
$$= \oint \vec{P}\cdot\hat{n}\, da - \int \nabla\cdot\vec{P}\, d\tau$$

$$= \oint \vec{P}\cdot d\vec{a} - \int \nabla\cdot\vec{P}\, d\tau$$

But $\oint \vec{P}\cdot d\vec{a} = \int \nabla\cdot\vec{P}\, d\tau$

by the divergence theorem so $Q_{tot} = 0$

4.15



$$\vec{P}(\vec{r}) = \frac{k}{r}\,\hat{r}$$

a) $\sigma_b = \vec{P}\cdot\hat{n}$

at $r = b$ $\quad \sigma_b = \frac{k}{b}$ since $\hat{n} = \hat{r}$

at $r = a$ $\quad \sigma_b = -\frac{k}{a}$ since $\hat{n} = -\hat{r}$

So the total surface for $r = a$, $r = b$ is

$$Q_{surface} = 4\pi b^2 \frac{k}{b} - 4\pi a^2 \frac{k}{a} = 4\pi k(b-a)$$

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2}$$

The total volume charge is

$$Q_{volume} = -4\pi \int_a^b r^2\left(\frac{k}{r^2}\right) dr = -4\pi k(b-a)$$

Note: $Q_{tot} = Q_{volume} + Q_{surface} = 0$

for $r<a$ $E=0$ (no charge).

for $a<r<b$ use Gauss's Law

$$\oint \bar{E}\cdot d\bar{a} = \frac{Q}{\varepsilon_0}$$

Q is the charge on the inner surface plus the charge out to a radius r in the volume. So:

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\left[-4\pi ha - 4\pi\int_a^r dr'\, r'^2\left(\frac{h}{r'^2}\right)\right]$$

$$= \frac{1}{\varepsilon_0}\left(-4\pi ha - 4\pi h(r-a)\right)$$

$$= \frac{1}{\varepsilon_0}\left(-4\pi h r\right)$$

$$E = -\frac{h}{\varepsilon_0 r} \qquad a<r<b$$

$E=0$ for $r>b$ since total charge $=0$

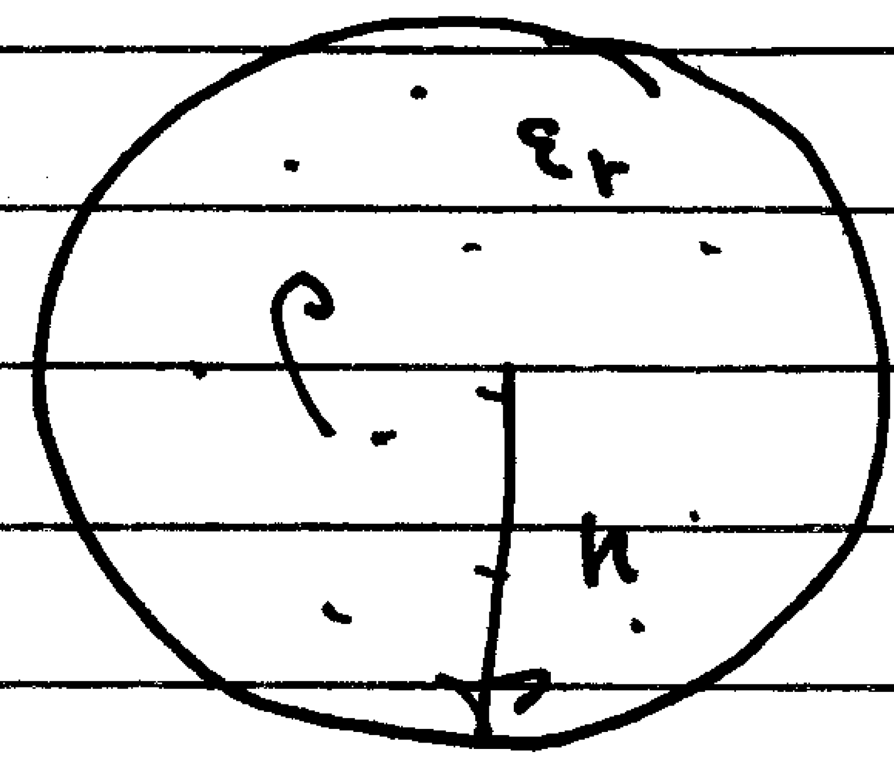
b) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0$ since there is no free charge

So for $r < a$ $\vec{P} = 0 \Rightarrow \vec{E} = 0$

for $a < r < b$ $\vec{E} = -\frac{\vec{P}}{\varepsilon_0} = -\frac{k}{\varepsilon_0 r}\hat{r}$

Fo $r > b$ $E = 0$ since $\vec{P} = 0$

4.20)



$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$ ρ = free charge density

$\vec{D} = \varepsilon \vec{E}$

$\oint \vec{D} \cdot d\vec{a} = Q \qquad D\, 4\pi r^2 = \rho \frac{4}{3}\pi r^3$

$D = \frac{\rho r}{3}$ for $r < R$

$E = \frac{\rho r}{3\varepsilon}$ " "

For $r > R$, we have

$$D = \frac{\rho \frac{4}{3}\pi R^3}{4\pi r^2} = \frac{\rho R^3}{3 r^2} \qquad r > R$$

$$E = \frac{\rho R^3}{3\varepsilon_0 r^2}$$

$r > R \qquad V(r) = \frac{\rho R^3}{3\varepsilon_0 r}$ since $E = -\frac{\partial V}{\partial r}$

$r < R \qquad V(r) = -\frac{\rho r^2}{6\varepsilon} + V_0$

Choose V_0 so that $V(r)$ is continuous at $r=R$

$$\frac{\rho R^2}{3\varepsilon_0} = -\frac{\rho R^2}{6\varepsilon} + V_0$$

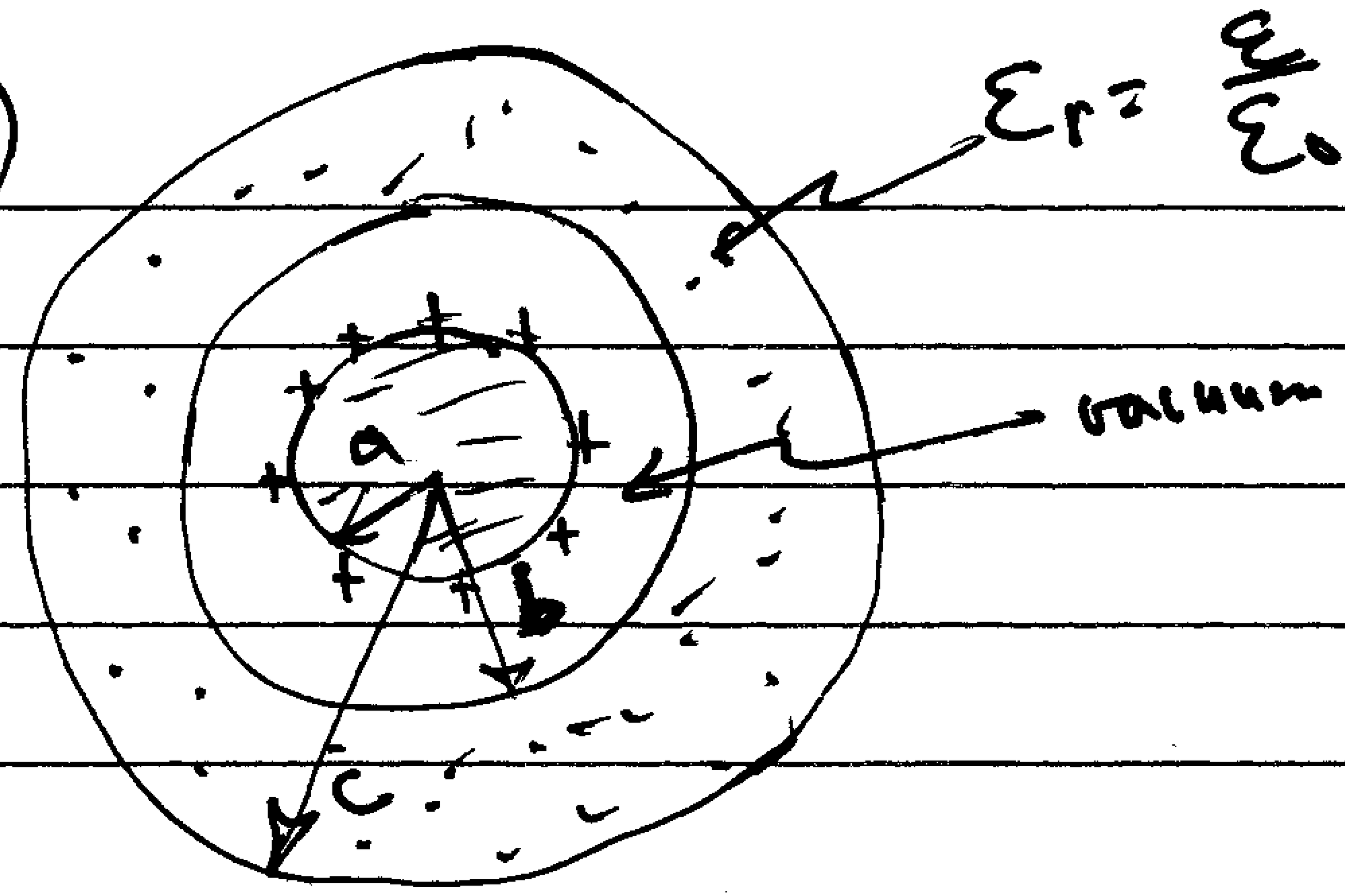
$$V_0 = \frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2}\frac{\varepsilon_0}{\varepsilon}\right)$$

So for $r<R$ $\quad V(r) = -\frac{\rho r^2}{6\varepsilon} + V_0$

and $\quad V(0) = V_0 = \frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2\varepsilon_r}\right)$

Note also that for $r>R$ $V(r) \to 0$ as $r \to \infty$.

4.21)



$$\oint \bar{D}\cdot d\bar{a} = q_f$$

Let l = length of the cylinder, then

$$D(2\pi s l) = \lambda l \qquad D = \frac{\lambda}{2\pi s}$$

$$E = \frac{D}{\varepsilon_0} = \frac{\lambda}{2\pi\varepsilon_0 s} \qquad b<r<a$$

$$E = \frac{D}{\varepsilon} = \frac{\lambda}{2\pi s \varepsilon} \qquad b<r<c$$

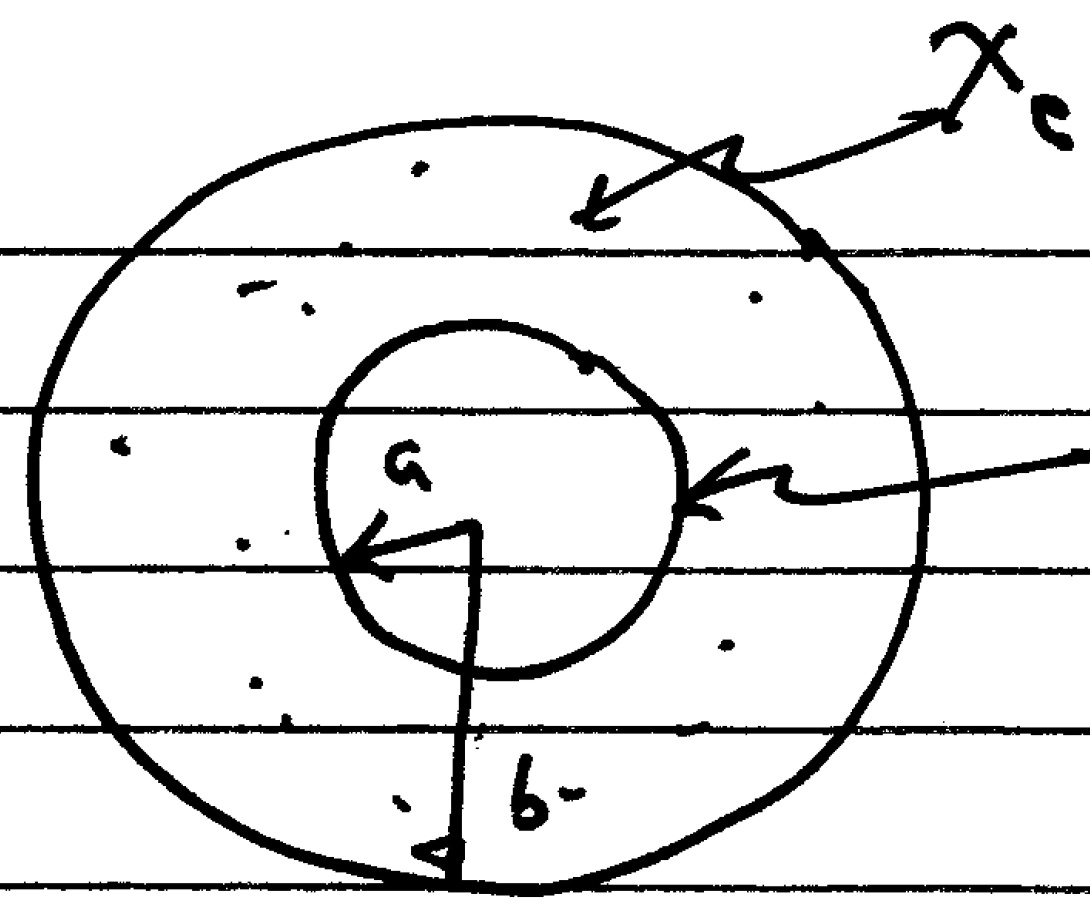
$$\Delta V = V_a - V_c = \int_a^c \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{\lambda}{2\pi\varepsilon}\ln\left(\frac{c}{b}\right)$$

$$\Delta V = \frac{q}{C} = \frac{\lambda l}{C} \qquad \frac{C}{l} = \frac{\lambda}{\Delta V}$$

$$\frac{C}{l} = \frac{1}{\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) + \frac{1}{2\pi\varepsilon}\ln(c/b)}.$$

$$= \frac{2\pi\varepsilon_0}{\ln(b/a) + \frac{1}{\varepsilon_r}\ln(c/b)}$$

4.26



$$W = \frac{1}{2}\int d\tau\, \vec{D}\cdot\vec{E}$$

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

For $r < a$ $E = 0$ $D = 0$ since all charge must be on the conductor surface

For $r > b$ $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$ $D = \varepsilon_0 E = \dfrac{Q}{4\pi r^2}$

For $a < r < b$ $D = \dfrac{Q}{4\pi r^2}$ $E = \dfrac{Q}{4\pi\varepsilon r^2}$

$$W = \frac{1}{2}(4\pi)\int_a^b dr\, r^2 \frac{Q^2}{(4\pi)^2\varepsilon}\frac{1}{r^4} + \frac{1}{2}(4\pi)\int_b^\infty dr\, r^2 \frac{Q^2}{(4\pi)^2\varepsilon_0 r^4}$$

$$= \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{b}\right)$$

$$\varepsilon = \varepsilon_0(1+\chi)$$

$$W = \frac{Q^2}{8\pi\varepsilon_0(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{b}\right)$$

$$W = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} - \frac{1}{b} + \frac{1+\chi_e}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} + \frac{\chi_e}{b}\right]$$